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Rational Chebyshev Approximation on Subsets

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In this note we study the closeness of a best Chebyshev approximation by generalized rational functions on a set, to a corresponding best approximation on a subset thereof. Such a problem is of practical interest, as we often determine a best approximation on an interval or rectangle in 2-space, as a limit of best approximations on finite point sets.

Let X be a compact set and let $\{\phi_1, \ldots, \phi_n\}$, $\{\psi_1, \ldots, \psi_m\}$ be linearly independent subsets of $C(X)$. Define

$$R(A, x) := P(A, x)/Q(A, x) := \sum_{k=1}^{n} a_k \phi_k(x) \Big/ \sum_{k=1}^{m} a_{n+k} \psi_k(x).$$

The conventions of Boehm (assuming his dense nonzero property is satisfied) or of Goldstein (stabilized rational functions) ([*2*], Chapter 9) can be used to give $R(A,x)$ a value when $Q(A,x) = 0$. For Y a compact subset of X, define

$$\|g\|_Y = \sup\{|g(x)| : x \in Y\},$$

$$\mathscr{R}(Y) = \{R(A,.) : \|R(A,.)\|_Y < \infty, \qquad Q(A,x) \geqslant 0 \quad \text{for} \quad x \in Y,\ Q(A,.) \not\equiv 0\}.$$

The problem of rational approximation of $f \in C(X)$ on Y, is to choose $r \in \mathscr{R}(Y)$ such that $\|f - r\|_Y$ is minimal. Such an r is called a best approximation to f on Y. It is a consequence of the theory of Boehm or Goldstein (depending on whose conventions we use) that a best approximation exists to all $f \in C(X)$ on Y.

Let $\{X_k\}$ be a given sequence of subsets of X such that for any $x \in X$, there is an $x_k \in X_k$ such that $\{x_k\} \to x$. Let $R(A_k, .)$ be a best approximation to f on X_k. We obtain in this note several results concerning convergence of $\{R(A_k, .)\}$ to a best approximation to f on X.

For convenience in existence and convergence arguments, we normalize rational functions $R(A,.)$ so that

$$\sum_{k=1}^{m} |a_{n+k}| = 1. \tag{1}$$

THEOREM 1. *Let r^* be best to f on X. Then*

$$\|f - R(A_k, .)\|_{X_k} \to \|f - r^*\|_X.$$

Proof. Define

$$\|A\| = \sum_{k=1}^{n} |a_k|.$$

Suppose that $\{\|A_k\|\}$ is an unbounded sequence. We can then assume without loss of generality, that $\{\|A_k\|\}$ is also a strictly increasing sequence. Define $B_k = A_k/\|A_k\|$; then $\{B_k\}$ is a bounded sequence with a limit point B. Select $z \in X$ such that $P(B,z) \neq 0$. Then there is a closed neighborhood N of z on which $P(B,.)$ does not vanish. Assume that $P(B,z) > 0$. Then

$$\eta = \inf\{P(B,x): x \in N\}$$

is positive and for all k sufficiently large, $P(B_k,x) > \eta/2$ for $x \in N$. It follows that $\inf\{P(A_k,x): x \in N\} \to \infty$ as $k \to \infty$. For all k sufficiently large we must have a point $x_k \in X_k \cap N$ such that

$$P(A_k,x_k) > 2\|f\|_X \sum_{k=1}^{m} \|\psi_k\|_X.$$

This implies that $R(A_k,x_k) > 2\|f\|_X$, which is impossible as

$$\|f - R(A_k,.)\|_{X_k} \geqslant |f(x) - R(A_k,x_k)| > \|f - 0\|_X,$$

contradicting $R(A_k,.)$ being best on X. It follows that $\{\|A_k\|\}$ is bounded (that is, the numerator coefficients of $\{R(A_k,.)\}$ are bounded). The denominator coefficients are bounded by the normalization (1).

$\{A_k\}$ being bounded, has a convergent subsequence, which we assume, without loss of generality, is $\{A_k\}$ itself, with limit A. We claim that $R(A,.)$ is a best approximation to f on X. Suppose the contrary; then there exists a point x and a positive ϵ such that

$$|f(x) - R(A,x)| > \|f - r^*\|_X + \epsilon. \tag{2}$$

The first possibility is that $Q(A,x) = 0$ and $P(A,x) \neq 0$. Let $\{x_k\} \to x, x_k \in X_k$; then $|R(A_k,x_k)| \to \infty$, which is impossible. The second possibility is that $Q(A,x) = P(A,x) = 0$. In the theory of Goldstein, $R(A,x)$ can always be defined in this case so that $f(x) - R(A,x) = 0$. In the theory of Boehm, we can find in the neighborhood of x a point at which $Q(A,.)$ does not vanish and for which an inequality of the type (2) holds. Thus, we need only consider the remaining possibility, which is that $Q(A,x) \neq 0$. In this case, let $\{x_k\} \to x, x_k \in X_k$; then $\{|f(x_k) - R(A_k,x_k)|\} \to |f(x) - R(A,x)|$ and for all k sufficiently large,

$$|f(x_k) - R(A_k,x_k)| > \|f - r^*\|_X + \epsilon.$$

This contradicts $R(A_k,.)$ being best to f on X_k. Thus, $R(A,.)$ is best and the theorem follows.

It is easily seen that $\{A_k\} \to A$, $Q(A,x) > 0$ imply that $R(A_k,x) \to R(A,x)$. An examination of the proof of the theorem gives

COROLLARY 1. *The sequence* $\{A_k\}$ *has a limit point. For any such limit point* A, $R(A,.)$ *is a best approximation to* f *on* X, *and a subsequence* $\{R(A_{k(j)},.)\}$ *of* $\{R(A_k,.)\}$ *exists converging pointwise to* $R(A,.)$ *outside the zeros of* $Q(A,.)$.

If $\{A_k\} \to A$ and $Q(A,.)$ has no zeros, $\{R(A_k,.)\}$ converges uniformly to $R(A,.)$. Again, an examination of the proof of Theorem 1 gives

COROLLARY 2. *Suppose* f *has a unique best approximation* $R(A,.)$ *which cannot be represented in the form* $R(C,.)$, *with* $Q(C,.)$ *having a zero in* X. *Then* $\{R(A_k,.)\}$ *converges uniformly to* $R(A,.)$ *on* X.

Define

$$S(A) = \{R(A,.)\, Q(B,.) + P(B,.)\}.$$

THEOREM 2. *Let* $R(A,.)$ *be best,* $Q(A,.) > 0$, *and* $S(A,.)$ *a Haar subspace. Then* $R(A,.)$ *is the unique best approximation.*

Proof. It is known ([*1*], p. 164) that $S(A,.)$ being a Haar subspace implies that $R(A,.) = p/q$ is the unique best approximation to f on X among rational functions with positive denominators. Suppose $R(C,.) = s/t$ is also a best approximation, t having a zero. Since

$$\frac{p(x)+s(x)}{q(x)+t(x)} = \begin{cases} \dfrac{q(x)}{q(x)+t(x)} \cdot \dfrac{p(x)}{q(x)} + \dfrac{t(x)}{q(x)+t(x)} \cdot \dfrac{s(x)}{t(x)} & \text{if } t(x) \neq 0, \\[2ex] \dfrac{p(x)}{q(x)} \quad \text{if } s(x) = t(x) = 0, \end{cases}$$

$(p+s)/(q+t)$ lies between p/q and s/t. It must therefore be also a best approximation. But it has a positive denominator, contradicting the uniqueness of p/q among such functions.

COROLLARY 3. *Let* f *have* $R(A,.)$ *as a best approximation,* $Q(A,.) > 0$, *and* $S(A,.)$ *be an* $(n+m-1)$*-dimensional Haar subspace. Then* $\{R(A_k,.)\}$ *converges uniformly to* $R(A,.)$ *on* X.

Proof. $S(A,.)$ is $(n+m-1)$-dimensional, $R(A,.)$ cannot be represented in the form $R(C,.)$, $Q(C,.)$ with a zero. By Theorem 2, $R(A,.)$ is the unique best approximation. The Corollary follows from Corollary 2.

The theory of this note can be applied to the classical problem of approximation by ratios of polynomials of degree $n-1$ to polynomials of degree $m-1$, on an interval $[a,b]$. Such ratios are customarily restricted to have positive denominators on $[a,b]$; but as any ratio of polynomials bounded on $[a,b]$ can be written in this form, this is no restriction. Assume without loss of generality that $Q(A,.) > 0$ and that $P(A,.)$, $Q(A,.)$ are relatively prime. Let $P(A,.)$, $Q(A,.)$ have exact degrees i, j, respectively. The subspace $S(A)$ is the space of ratios of polynomials of degree $n+m-2-\min\{n-1-i, m-1-j\}$ to $Q(A,.)$, a Haar subspace of dimension $n+m-1-\min\{n-1-i, m-1-j\}$. The hypotheses of Corollary 3 are satisfied if and only if $i = n-1$ or $j = m-1$, that is $R(A,.)$ is nondegenerate, or, equivalently, of maximum degree (in the sense of Rice).

We give an example to show that $\{R(A_k,.)\}$ need not converge uniformly to $R(A,.)$ even if $R(A_k,.)$ is a unique best approximation. Let $X = [-1,1]$, $f(x) = x$, and $R(A,x) = a_1/(a_2 + a_3 x)$. As f alternates once on $[-1,1]$, 0 is the unique best approximation to f. Now let $X_k = [-1 + 1/k, 1]$; then f does not alternate once on X_k. Since all elements of $\mathscr{R}(X_k)$ except the zero element are of degree 2, the unique best approximation r_k to f on X_k is characterized by $f - r_k$ alternating at least twice on X_k. If $\{r_k\}$ converged uniformly to 0, f would have to alternate twice. By drawing a diagram it can be seen that $r_k(-1)$ does not converge to zero. This example shows that the dimensionality condition of Corollaries 2 and 3 cannot be deleted.

References

1. E. W. Cheney, "Introduction to Approximation Theory." McGraw-Hill, New York, 1966.
2. J. Rice, "The Approximation of Functions," Volume 2. Addison-Wesley, Reading, 1969.